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The Density of States for Almost Periodic Schrödinger Operators and the Frequency Module: A Counter-Example

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Abstract. We exhibit an example of a one-dimensional discrete Schrödinger operator with almost periodic potential for which the steps of the density of states do not belong to the frequency module. This example is suggested by the K-theory [3].

Introduction

The problem of investigating the spectrum of quantum almost periodic hamiltonian operators has increased very recently in importance due to new information obtained by several authors.

Among these progresses, the integrated density of states $\mathfrak{N}(E)$ has been interpreted in the algebraic framework [9, 3]: the trace of the spectral measure associated with the random hamiltonian as an element of the canonically associated von Neumann algebra [2]. If the energy belongs to the resolvent set, where $\mathfrak{N}(E)$ is locally constant, the density of states takes values in the K_0-group (precisely in its image by the trace) of the canonical C^*-algebra constructed from the quasi periodic hamiltonian.

In the case of a one-dimensional Schrödinger operator with an almost periodic potential V, this group coïncides with the frequency module of V [6, 3]. In this short note, we exhibit an example of a one-dimensional Schrödinger operator with a "discontinuous quasi-periodic" potential for which the K-group is different from the frequency module, and we show that the values of $\mathfrak{N}(E)$ at the steps are really not in the frequency module.

To be precise we deal with a hamiltonian $(H_x)_{x\in\mathbb{T}}$ acting on $\ell^2(\mathbb{Z})$ by

$$H_x\psi(n)=\psi(n+1)+\psi(n-1)+V(x-n\theta)\psi(n), \tag{I.1}$$

where $V\in\mathscr{C}(\mathbb{T})$ and θ is an irrational number. The spectral density in this case is defined by

$$\mathfrak{N}(E)=\lim_{N\to\infty}(2N+1)^{-1}\,\mathrm{card}\{\text{eigenvalues of } H_x\restriction_{\ell^2(-N,N)}<E\}\,. \tag{I.2}$$

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