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Linking Numbers of Closed Manifolds at Random in $\mathbb{R}^n$, Inductances and Contacts

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Abstract. We give here new results of topology and integral geometry concerning the Gauss linking number I of closed manifolds in n-dimensional space. The rigid manifolds have arbitrary shapes and dimensions, and are statistically at random positions in $\mathbb{R}^n$. Generalizing Pohl's work, for two closed manifolds $\mathscr{C}_1^r$, $\mathscr{C}_2^s$, of respective dimensions r and s, with $0 \leqq r \leqq n-1$, and $r+s+1=n$, we consider the "kinematic linking integral" $\mathscr{I} = \langle \int I^2(\mathbf{x}, \mathcal{O}) d^n x \rangle$, of the square linking number I of $\mathscr{C}_1^r$ and $\mathscr{C}_2^s$, over the group of Euclidean motions of one manifold (translations $\mathbf{x}$, rotations $\mathcal{O}$). Introducing a new tensorial method, and using group theory, we show quite generally that $\mathscr{I} = \text{num. fact.} \int_0^\infty d\varrho [\mathscr{A}_1(\varrho)\mathscr{A}_2(\varrho) + \delta_{r,s}\mathscr{B}_1(\varrho)\mathscr{B}_2(\varrho)]$, where ϱ is a length variable and where $\mathscr{A}_\alpha$, $\mathscr{B}_\alpha (\alpha = 1, 2)$ are characteristic functions associated with the manifold $\mathscr{C}_\alpha$ only. We study functions $\mathscr{A}$ and $\mathscr{B}$ of a manifold $\mathscr{C}^r$, of dimension r, in all cases $0 \leqq r \leqq n-1$. $\mathscr{A}$ always exists. $\mathscr{A}(0)$ gives $\mathscr{C}$'s area, whereas $\int_0^\infty \mathscr{A}(\varrho) d\varrho$ equals the interior volume of a hypersurface $\mathscr{C}$. $\mathscr{B}$ is found to exist and not to vanish only if $2 \dim \mathscr{C} + 1 = n$ and $n = 3 + 4q = 3, 7, 11 \ldots$ $\mathscr{A}$ and $\mathscr{B}$ are explicitly calculated for segments and r-spheres S^r. As an application the topological excluded volume of a gas of nonlinked spheres S^r moving in $\mathbb{R}^{2r+1}$ is calculated. We generalize to N manifolds $\mathscr{C}_\alpha$, $\alpha = 1, \ldots, N$, linked successively to each other and forming a ring. The cyclic product of their linking numbers is integrated over the group of motions of the manifolds. It is shown to factorize completely in Fourier space, with special algebraic rules, over the set of $2N$ characteristic functions $\mathscr{A}_\alpha$, $\mathscr{B}_\alpha$, associated with the $\mathscr{C}_\alpha$'s. The same algebra of characteristic functions is shown to describe a larger class of topology and electromagnetism properties: a new theorem is given for a family of Euclidean group integrals involving the random linking numbers, mutual inductances and contact distributions of N manifolds.

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