Commun. Math. Phys. 85, 177–196 (1982)

Communications in
Mathematical
Physics
© Springer-Verlag 1982

The Low Energy Scattering for Slowly Decreasing Potentials

D. R. Yafaev

Leningrad Department of Mathematical Institute, 27, r. Fontanka, 191011, Leningrad, USSR

Abstract. For the radial Schrödinger equation with a potential $q(x)$ decreasing at infinity as $q_0 x^{-\alpha}$, $\alpha \in (0, 2)$, the low energy asymptotics of spectral and scattering data is found. In particular, it is shown that for $q_0 > 0$ the spectral function vanishes exponentially as the energy k^2 tends to zero. On the contrary, there is always a zero-energy resonance for $q_0 < 0$. These results determine the local asymptotics of solutions of the time-dependent Schrödinger equation for large times t. Specifically, for positive potentials its solutions decay as $\exp(-\vartheta_0 t^{(2-\alpha)/(2+\alpha)}$, $\vartheta_0 > 0, t \to \infty$. In the case $\alpha \in (1,2)$ it is shown that for $\pm q_0 > 0$ the phase shift tends to $\pm\infty$ as $k \to 0$ and its asymptotics is evaluated.

1. Introduction

In the study of the low energy scattering of nonrelativistic particles it is usually assumed that the potential is vanishing sufficiently quickly at infinity. Moreover, in the physics literature potentials are often assumed to be central. In this case the low energy scattering is determined [1, 2] only by the behaviour of particles with a zero angular momentum ℓ. For $\ell = 0$ the phase shift and generically the partial cross section have finite limits as the energy tends to zero. This shows that scattering at low energies depends weakly on the shape of the potential. Specifically, low energy scattering is well described in terms of the scattering length. Recently the low energy asymptotics of scattering data was found and rigorously proved by A. Jensen and T. Kato [3] for arbitrary quickly decreasing potentials. The behaviour at low energies of a spectral family of the corresponding Hamiltonian was also investigated in [3]. Hence the local decay in time of the solution of the time-dependent Schrödinger equation is deduced. Namely, the solution decays [4, 5, 3] generically as $t^{-3/2}$ if the initial state is sufficiently localized in the space and is orthogonal to the bound states. Most of these results are valid if $q(\mathbf{x}) = O(|\mathbf{x}|^{-2-\varepsilon})$, $\varepsilon > 0$. We call such potentials quickly decreasing or short-range.

In this paper it is found that for potentials vanishing slower than $|\mathbf{x}|^{-2}$ as $|\mathbf{x}| \to \infty$, the low energy behaviour of spectral and scattering data is essentially different

0010-3616/82/0085/0177/$04.00