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On A. Connes' Noncommutative Integration Theory*

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Abstract. This paper is a comment (written in a self-contained way) upon A. Connes' noncommutative integration theory. A simple observation (Lemma 9) allows the direct definition of transverse measures as functionals on the general positive random functions. Connes' first definition is then recovered by considering the transverse functions as a separating subset of the positive random functions (to which they are related by explicit natural transformations).

Introduction

The aim of this paper is to provide easier (at any rate more leisurely) access to the notion of transverse measures on groupoids developed in [1], which we believe to be connected with quantization (we hope to return to this later). Alain Connes was led to his noncommutative integration theory in his generalization of the Index Theorem to foliations on compact manifolds, by the need to integrate transversally an everywhere infinite function (the dimension of the space of harmonic forms on the generically noncompact leaves). The two major points of this theory are the following: on the one hand, it allows integration over singular quotients (e.g., non standard orbit spaces of ergodic actions), giving a precise meaning to a heuristic situation where the space shrinks whilst the function becomes infinite (tiny space). On the other, it provides the right (i.e., fully functorial) description of KMS states of convolution algebras of groupoids, these being a possibly largely universal model for noncommutative algebras, related with "geometry" via the groupoid. Consideration of the tiny space in fact enforces noncommutativity — a feature reflecting the incorporation to classical analysis of the non type I phenomena encountered in operator algebras (and in the algebraic theory of infinite quantum systems).

One of the basic ideas of Connes' integration theory is a generalization of the notion of a (real- or integer-valued) numerical function. Instead of plotting a number against the variable, one plots a standard Borel set with "cardinality"

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