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On the Approach to the Final Aperiodic Regime in Maps of the Interval

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Abstract. We present a global approach to the final aperiodic regime in maps of the interval displaying a simple pattern with similarities to the Feigenbaum scheme.

1. Introduction

It is well known that complex dynamical behaviour can be generated by maps of the interval into itself. In particular the mechanism of period doubling bifurcation by functional iteration provides a procedure to achieve, starting with simple periodic systems, aperiodic regimes. For functions $f_\mu(x)$ belonging to $C^2[-1,1]$, even and convex, depending on a parameter μ in $[0,1]$ such that (for every x) $f_0(x)=1$, $f_1(0)=-1$ and $f_\mu(\pm 1)=1$, numerical and analytic studies have revealed remarkable properties in the approach of the aperiodic regime by period doubling bifurcation. Let us briefly summarize here these properties [1]:

i) Each orbit of period 2^n has in μ a region I_n of stability above which it ceases to be stable originating (by bifurcation) an orbit of period 2^{n+1}.

ii) The sequence (μ_n), $n=0,1,2,\ldots$, where μ_n is the value of μ for which the orbit of period 2^n is super-stable, is strictly increasing with limit $\mu_\infty<1$.

iii) As $n\to\infty$ the sequence $(\mu_n-\mu_{n-1})$ converges geometrically, i.e.,

$$\lim \frac{\mu_n-\mu_{n-1}}{\mu_{n+1}-\mu_n}=\delta.$$

iv) The Feigenbaunm ratio δ only depends on the behaviour of $f_\mu(x)$ at the critical point $x=0$ (universality).

v) The iterated functions $f^{2^n}_{\mu_n}(x)$ and $f^{2^{n+1}}_{\mu_{n+1}}(x)$ are similar and related by a scaling transformation which determines a second Feigenbaum ratio α.

In the present paper we consider the approach to aperiodic behaviour from a more global point of view. After the first Feigenbaum window in μ, $[0,\mu_\infty]$, other windows of periods $N2^n$, $N=3,\ldots$, $n=0,1,\ldots$, as well as regions of aperiodicity follow. The final aperiodic limit is achieved for $\mu\to 1$. The various regions of stability for the different periods, and the aperiodic regions are all mixed up in a

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