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On Witten's Proof of the Positive Energy Theorem*

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Abstract. This paper gives a mathematically rigorous proof of the positive energy theorem using spinors. This completes and simplifies the original argument presented by Edward Witten. We clarify the geometric aspects of this argument and prove the necessary analytic theorems concerning the relevant Dirac operator.

The positive energy theorem in general relativity states that an isolated gravitational system with nonnegative local matter density must have nonnegative total energy, measured at spatial infinity. This was originally conjectured more than twenty years ago by Arnowitt, Deser and Misner [1]. Subsequently, a great many people worked on this problem and proved various special cases. Finally, the generic case was established by Schoen and Yau [10–13]. For a history of the problem, with complete references, we refer the reader to the papers of Geroch [5] and Witten [15].

Recently E. Witten has presented a simple new argument for a proof of the positive energy theorem [15]. However, several points of his argument require justification. This paper gives a complete, rigorous and self-contained proof of the positive energy theorem, based on Witten's formulation. In addition to supplying the necessary analytic theorems, we present the proof in its geometric context.

The first three sections present the background for Witten's proof in the language of differential geometry. This involves a brief discussion of Dirac spinors the definition of the Dirac operator along a spacelike hypersurface, and a derivation of Witten's formula for the square of this operator. Section four contains the statement of our main result: the existence of a Green's function for the hypersurface Dirac operator. The positive energy theorem is then proved as a consequence of this fact. The estimates and analysis required for the construction of the Green's function are presented in Sect. five.

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