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On the Regularization of the Kepler Problem

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Abstract. An intimate relationship between Moser's regularization [1] and the KS-regularization [3] of the 3-dimensional Kepler problem is established. Explicit formulae linking Moser's and the KS-transformation are obtained in the case of negative as well as in the case of positive energies. As a side result it is shown that the KS-transformation owes its existence to the local isomorphism of SO(2,4) and SU(2,2).

1. Introduction

In [1] (see also [2]) Moser, starting from a stereographic projection in configuration space, constructs a diffeomorphism that carries the geodesic flow on the unit tangent bundle of the pointed n-sphere onto the flow of the n-dimensional Kepler problem on a surface of fixed negative energy. The missing point together with an $(n-1)$-sphere of directions correspond to the collision states of the Kepler problem. When the Kepler flow on a surface of fixed negative energy is replaced by the geodesic flow on the unit tangent bundle of the n-sphere, the collision states are "regularized", i.e. they loose their exceptional status and are indistinguishable from all the other states. This "regularization" has the fringe benefit of exposing the hidden $SO(n+1)$-symmetry of the Kepler problem. This symmetry in turn makes it obvious that besides the $\frac{1}{2}n(n-1)$ components of the angular momentum integral, the Kepler problem possesses n additional integrals which together make up the Lenz-Runge vector (see in particular [2]).

A seemingly quite different procedure which achieves a regularization of the Kepler problem was proposed by Kustaanheimo and Stiefel (KS) in [3]. Their procedure has been explained in great detail in the monograph [4]. It is based on the KS-transformation which generalizes the Levi Civita transformation from two to three dimensions. The KS-transformation replaces the 3-dimensional Kepler Hamiltonian (with fictitious time s) by a Hamiltonian of four harmonic oscillators in resonance-denoted by J in the sequel – whose energy surfaces are 7-spheres embedded $\mathbb{R}^8(=\mathbb{C}^4)$. However, only points that also lie on a certain seven-

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