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Monopoles and Geodesics

N. J. Hitchin

St. Catherine's College, Oxford OX1 3UJ, England

Abstract. Using the holomorphic geometry of the space of straight lines in Euclidean 3-space, it is shown that every static monopole of charge k may be constructed canonically from an algebraic curve by means of the Atiyah-Ward Ansatz $\mathscr{A}_k$.

1. Introduction

It has been known for some time that the Bogomolny equations, describing static Yang-Mills-Higgs monopoles in the Prasad-Sommerfield limit, may be solved by twistor methods. Indeed, they can be reinterpreted as the self-duality equations in Euclidean four-space which are in addition time-translation invariant, and the methods of Penrose, Ward and Atiyah may be applied directly. During the past year significant progress has been made using this line of attack by Ward [15, 16], Prasad and Rossi [12], and Corrigan and Goddard [7]. They all use a variant of the Atiyah-Ward $\mathscr{A}_k$-Ansatz [3] to construct an SU(2) monopole of charge k. The main purpose of this paper is to show that *every* solution of the Bogomolny equations satisfying the appropriate boundary conditions can be constructed in a canonical manner by this method.

Our approach is again twistorial, but instead of passing from a problem in 3-space to one in 4-space, we use complex methods intrinsically associated to the Euclidean geometry of $\mathbb{R}^3$. We replace the set of points of $\mathbb{R}^3$ by the space of oriented geodesics (straight lines). This has the structure of a complex surface (in fact, the holomorphic tangent bundle $\mathbf{T}$ to the projective line) and a solution to the Bogomolny equations gives rise in a natural manner to a holomorphic vector bundle over this surface. Actually, this approach to problems in Euclidean space is by no means new – it was used by Weierstrass in 1866 to solve the minimal surface equations.

Briefly, our method consists of defining a vector bundle $\tilde{E}$ over the surface $\mathbf{T}$ of geodesics by associating to each straight line the null space of the differential operator

$$D = \nabla_u - i\Phi\,,$$

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