Commun. Math. Phys. 83, 77–82 (1982)

Communications in
Mathematical
Physics

The Perturbation Series for Φ_3^4 Field Theory is Divergent

C. de Calan and V. Rivasseau

Centre de Physique Théorique de l'Ecole Polytechnique, Plateau de Palaiseau, F-91128 Palaiseau Cedex, France

Abstract. We prove in a rigorous way the statement of the title.

I. Introduction

At least since a paper by Dyson [1], the perturbation expansion for field theories like Φ^4 or Q.E.D. is commonly believed to be a divergent series. However it has been periodically noticed [2] [3] [4] that this statement is rigorously proved only for the simplest models, i.e. $[P(\Phi)]_2$ [5].

In the case of euclidean Φ_3^4, we know the results of constructive field theory [6] and the Borel summability of the perturbation expansion [7]. Yet there is no proof that this series is not actually convergent. The difficulty which prevents Jaffe's method [5] from working for Φ_3^4 is the change in the signs of some amplitudes, due to the renormalization, which could produce cancellations at each order. We solve this problem by rewriting the usual perturbation expansion in terms of convergent positive amplitudes involving a "dressed" propagator. This method is an iteration of the procedure used by Hepp [3] for the regularized version of the model.

More generally, the control of signs in renormalized perturbation series could allow one to go beyond the recent results on the Borel transformed series for Φ^4 theories [8] [9]. Extensions of our method might then provide rigorous results on the presence—or absence—of singularities on the real axis of the Borel plane ("instantons" or "renormalons" [10]).

II. Proof

We consider massive scalar bosons (the mass-scale is fixed by taking $m=1$) self-interacting via the lagrangian density $\mathscr{L}_I = -\frac{\lambda}{4!} : \Phi^4 :$ in an euclidean space-time with three dimensions. The formal series in λ defining a given N-points connected Schwinger function S_N is given in terms of Feynman graphs G (with $N(G)$ external lines, $n(G)$ vertices) by:

$$S_N(p,\lambda) = \sum_{\substack{G \\ N(G)=N}} A_G^R(p,\lambda), \tag{1}$$

0010-3616/82/0083/0077/$01.20