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Connections with L^p Bounds on Curvature

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Abstract. We show by means of the implicit function theorem that Coulomb gauges exist for fields over a ball in R^n when the integral $L^{n/2}$ field norm is sufficiently small. We then are able to prove a weak compactness theorem for fields on compact manifolds with L^p integral norms bounded, $p > n/2$.

Introduction

The variational problems for gauge fields arising in physics differ markedly from many other geometric variational problems due to their gauge invariance. This paper provides two technical tools for handling the gauge invariance. First we show the local existence of a "good" gauge (called Lorentz, Hodge or Coulomb) under very weak hypotheses. Secondly, we prove a global theorem on the weak compactness of connections given integral bounds on their curvatures. These technical theorems are very useful for both regularity theorems and direct variational methods. I am particularly indebted to C. Taubes, who pointed out some very important generalizations of the original theorems. The strong form $2p = n$ of Corollary 1.4 and Corollary 2.2 are essentially due to Taubes.

In Sect. 1, we present notation and state the theorems and a few immediate applications. Detailed proofs are in Sect. 2 for the local results, and in Sect. 3 for the global results.

1. Notation and Statement of the Results

In this paper, η is a vector bundle with compact structure group G over a compact Riemannian n-dimensional manifold M. Assume the fibers $\eta_x \cong R^\ell$ carry an inner product and that $G \subset SO(\ell)$ respects this inner product. The bundle Aut η is the automorphism bundle with fiber $(\mathrm{Aut}\,\eta)_x \cong G$. The bundle Ad η is the Lie algebra or adjoint bundle with fiber $(\mathrm{Ad}\,\eta)_x \cong \mathfrak{G}$, the Lie albebra of G. Assume the metric on Aut η and Ad η are compatible with the usual metric on $SO(\ell)$.

Let $\mathfrak{A}$ be the space of smooth connections on η compatible with the structure group. Every such connection D induces a connection on the Ad η bundle which is also called D. In this case we have the Riemannian connection on the tangent bundle TM; therefore $D \in \mathfrak{A}$ induces connections on all bundles associated to

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