Commun. Math. Phys. 82, 471–495 (1982)

Communications in Mathematical Physics

The Spectral Class of the Quantum-Mechanical Harmonic Oscillator

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Abstract. The purpose of this paper is to study the so-called *spectral class* **Q** of anharmonic oscillators $Q = -D^2 + q$ having the same spectrum $\lambda_n = 2n$ $(n \geqq 0)$ as the harmonic oscillator $Q^0 = -D^2 + x^2 - 1$. The *norming constants* $t_n = \lim_{x\uparrow\infty} \ell g\,[(-1)^n e_n(x)/e_n(-x)]$ of the eigenfunctions of Q form a complete set of coordinates in **Q** in terms of which the potential may be expressed as $q = x^2 - 1 - 2D^2\,\ell g\,\theta$ with

$$\theta = \det\left[\delta_{ij} + (e^{t_i} - 1)\int_x^\infty e_i^0 e_j^0 : 0 \leqq i, j, < \infty\right],$$

e_n^0 being the n^{th} eigenfunction Q^0. The spectrum and norming constants are canonically conjugate relative to the bracket $[F, G] = \int \nabla F D \nabla G dx$, *to wit*: $[\lambda_i, \lambda_j] = 0$, $[t_i, 2\lambda_j] = 1$ or 0 according to whether $i = j$ or not, and $[t_i, t_j] = 0$. This prompts an investigation of the symplectic geometry of **Q**. The function θ is related to the theta function of a singular algebraic curve. Numerical results are also presented.

1. Introduction

The spectrum of the quantum-mechanical harmonic oscillator[1] $Q^0 = -D^2 + x^2 - 1$ is 0, 2, 4, 6, etc. The corresponding unit eigenfunctions are the Hermite functions:

$$e_n^0(x) = (\sqrt{\pi}2^n n!)^{-1/2} e^{x^2/2} D^n e^{-x^2} \qquad (n \geqq 0).$$

Let Δq belong to the class **S(R)** of real infinitely differentiable functions vanishing rapidly at $\pm\infty$[2]. The anharmonic oscillator $Q = -D^2 + q$ with potential $q = x^2 - 1 + \Delta q$ has a discrete spectrum of simple eigenvalues $\lambda_n = \lambda_n[q]$, increasing to $+\infty$ with n, and corresponding unit eigenfunctions e_n $(n \geqq 0)$ of class **S**. The

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1 D signifies differentiation with regard to x.

2 $x^i D^j \Delta q = o(1)$ for $x \to \pm\infty$ and every $i, j \geqq 0$.

0010-3616/82/0082/0471/$05.00