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Integrable Nonlinear Equations and Liouville's Theorem, I

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Abstract. A symplectic structure for stationary Lax equations of the type $[L, P] = 0$ is constructed, where L is a matrix differential operator of the first order. It is shown that the equation has a sufficient for the complete integrability amount of first integrals in involution. The well-known linearization of the equation by the Abelian mapping is obtained in a natural manner in consequent exercising of Liouville's procedure.

This paper continues a series of works by Gelfand and the author ([2–4] and others) which deal with the equations of the Lax type $L_t = [P, L]$. Here P, L are differential operators such that the order of $[P, L]$ is less than that of L. Novikov noticed that the stationary variants of these Lax equations are equally interesting. These stationary variants are obtained by assuming that P, L are independent of t, i.e. equations of the form $[P, L] = 0$ (sometimes called the Novikov equations). They are ordinary differential equations, totally integrable in a pure classical sense, i.e. solvable by quadratures. Since then the theory has branched into two parts: **nonstationary** and **stationary.** They often come into contact. For example, if there are two commuting Lax equations (e.g. two higher KdV equations) then the set of all solutions of the stationary problem for one of them is an invariant (and finite dimensional) manifold for the other. In such a way solitons and periodic finite-zonal solutions can be obtained. Nevertheless the nonstationary theory and the stationary one are quite different, with different problems and methods.

Gelfand and the author in their joint work were guided by an idea that it is the Hamiltonian structure of the equations under consideration which plays the leading role. The fact that the KdV equation is one of the Hamiltonian types was first established by Gardner, Zakharov and Faddeev; Gelfand and the author have given the general construction of Lax equations for an arbitrary order of the operator L, scalar as well as matrix (another construction was given by Kritchever) and proved that these equations are Hamiltonian. Moreover, they have constructed explicitly the Hamiltonian structure of the equations. Since then this structure has been studied by many authors. The greatest progress was achieved by Adler, Manin and Lebedev who have given a transparent group explanation of this Hamiltonian structure. Let us also mention works by Kuperschmidt and Wilson,

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