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Local Existence of the Borel Transform in Euclidean Φ_4^4

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Abstract. We bound rigorously the large order behaviour of Φ_4^4 euclidean perturbative quantum field theory, as the simplest example of renormalizable, but non-super-renormalizable theory. The needed methods are developed to take into account the structure of renormalization, which plays a crucial role in the estimates. As a main theorem, it is shown that the Schwinger functions at order n are bounded by $K^n n!$, which implies a finite radius of convergence for the Borel transform of the perturbation series.

I. Introduction

In this work we give rigorous bounds on the Feynman amplitudes at large order for renormalized Φ_4^4 euclidean quantum field theory. We prove the "local existence" of the Borel transform of its perturbative series for the connected Green's functions (or Schwinger functions): the Borel transformed perturbative series is shown to have a finite radius of convergence. Such a theorem is reached by finding direct estimates of the renormalized amplitudes related to Feynman graphs. We combine the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization scheme with the use of the so-called Hepp sectors to give bounds which depend on the renormalization structure of the graphs. Then we count the number of graphs with a given renormalization structure, and we can bound the term of order n of any Schwinger function by $K^n n!$, which proves our main result.

The same property was already known for Φ_ν^4 in the super-renormalizable domain $\mathrm{Re}\,\nu<4$ [1]. Through Borel summability for integer dimensions $\nu=2,3$ [2], perturbative and nonperturbative methods of constructive quantum field theory are connected [3]. In this domain, each Feynman amplitude is uniformly bounded by K^n. Conversely for $\nu=4$, such a bound cannot exist and the situation remains unresolved. Despite all efforts there has been almost no rigorous result. It has been remarked that some graphs of order n grow like $n!$ [4]. (Of course we recover this particular result and we show how few graphs are dangerous.) Heuristic arguments have been given, to make plausible the appearance of singularities, called renormalons, on the real positive axis of the Borel plane [5].

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