Commun. Math. Phys. 81, 437–453 (1981)

Communications in
Mathematical
Physics

Analyticity Properties of the Feigenbaum Function

H. Epstein[1] and J. Lascoux[2]

1 Institut des Hautes Etudes Scientifiques, F-91440 Bures-sur-Yvette, France
2 Centre de Physique Théorique, Ecole Polytechnique, F-91120 Palaiseau, France

Abstract. Analyticity properties of the Feigenbaum function [a solution of $g(x) = -\lambda^{-1}g(g(\lambda x))$ with $g(0)=1, g'(0)=0, g''(0)<0$] are investigated by studying its inverse function which turns out to be Herglotz or anti-Herglotz on all its sheets. It is found that g is analytic and uniform in a domain with a natural boundary.

0. Introduction

In the theory of successive period doublings of one-parameter families of smooth mappings of the interval $[-1, 1]$, an important role is played by one particular such function, here denoted g, which is a solution of a certain functional equation [see Eq. (1) below]. This theory is expounded at length in references [5, 6, 2–4, 8] and will not be recalled here. The purpose of this note is to indicate a few analyticity properties of this function, which might, in the future, throw some light on the still somewhat mysterious aspects of this theory.

Proofs of the existence of g have been provided successively by Lanford [7, 9], Campanino et al. [1], and again by Lanford [10]. None of them is truly satisfactory (see comments in [8]).

0.1. Notations

We denote $\Pi_+ = -\Pi_- = \{\zeta \in \mathbb{C} : \mathrm{Im}\,\zeta > 0\}$ the open upper half plane. $\bar{\zeta}$ will always denote the complex conjugate of $\zeta \in \mathbb{C}$, and, to avoid confusions, the closure of a set E will be denoted E^c. A holomorphic function φ of a complex variable is called "self-conjugate" if $\varphi(\zeta) = \bar{\varphi}(\bar{\zeta})$.

0010-3616/81/0081/0437/$03.40