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Estimates on the Vorticity of Solutions to the Navier-Stokes Equations*

Vladimir Scheffer

Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

Abstract. We estimate the vorticity of the flow of an incompressible, viscous, three dimensional fluid near the boundary of its container. We obtain a bound that is valid outside a small subset of space-time with special properties.

1. Introduction

It is the basis of the Prandtl boundary layer theory that vorticity is introduced into solutions of the Navier-Stokes equations through a boundary layer. Therefore, it is important to obtain estimates of the size of the vorticity close to the boundary. The theorem below yields the following type of estimate: We fix a small positive number τ and examine points (x, t) in space-time, where x lies at a distance τ from the boundary. We also assume that the time elapsed since the beginning of the flow is at least τ^2. Then the size of the vorticity at (x, t) is at most $O(\tau^{-2})$ unless (x, t) happens to lie in a certain set. This set is the union of cylinders of size τ. The number of different cylinders is at most $O(\tau^{-5/3})$. Since the cylinders are subsets of space-time, their union is a small set. However, the important point is not the measure of this set. The interesting thing is the clustering of this set into lumps of size τ. Outside of these lumps we have uniform estimates on the vorticity.

The proof of the main theorem in [1] involved the construction of a similar set. There the set was the union of finitely many cylinders A_i where $\sum_i$ (diameter of $A_i)^{5/3}$ was bounded by a constant that depended only on the initial kinetic energy. In addition, the maximum of the diameters of the A_i could be made arbitrarily small. The theorem below is an improvement on this.

One can go further and state that the vorticity is continuous at the points where we can estimate its size. This is a consequence of the local boundedness

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