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Surface Integrals and Monopole Charges in Non-Abelian Gauge Theories*

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Abstract. We derive a formula which gives all the magnetic charges (topological invariants) of a monopole in the adjoint representation of a non-abelian gauge theory in terms of surface integrals at infinity.

1. Introduction

It has been known for some time that there exist topological invariants which are associated with static Yang–Mills–Higgs field configurations on Minkowski space [1–3]. In particular, suppose that the gauge group G is a simple, compact Lie group. Further, assume that the Higgs field is in the adjoint representation of the Lie algebra $\mathscr{g}$ of G. Every field configuration satisfying certain asymptotic conditions (c.f. Theorem 2.1) is known to define a gauge invariant set of integers $\{n_a\}_{a=1}^{\ell}$, $\ell \leqq \operatorname{rank} G$ [3]. These integers are the aforementioned topological invariants. It is the purpose of this paper to prove that the integers $\{n_a\}_{a=1}^{\ell}$ are completely specified by surface integrals at $|x| = \infty$. For example, if $G = SU(n)$ and the representation of $\mathscr{g}$ is the defining one, then

$$Q_k = \lim_{R\to\infty} \frac{1}{4\pi} \int_{|x|=R} \operatorname{tr}(\Phi^k F_A), \qquad k \in \{1, \ldots, \ell\} \tag{1.1}$$

completely determine the integers $\{n_a\}_{a=1}^{\ell}$. Here Φ is the Higgs field; the Lie algebra-valued two form F_A is the curvature of the Yang–Mills connection A; and $x = (x^1, x^2, x^3)$ are cartesian coordinates on $\mathbb{R}^3$. For example, if $G = SU(2)$ then only Q_1 is needed. In this case the right hand side of (1.1) computes the winding number of the map

$$\hat{\Phi} = \Phi/|\Phi| : S_R^2 = \{x \in \mathbb{R}^3 : |x| = R\} \to S^2 = \{\sigma \in \mathscr{su}(2) : |\sigma| = 1\}\ [1]$$

(see also [4], Proposition II.3.7.)

We remark that the right hand side of (1.1) is gauge invariant so it is not surprising that there should be some connection between the numbers $\{Q_k\}_{a=1}^{\ell}$ and the integer invariants $\{n_k\}_{k=1}^{\ell}$. This relationship is stated as Theorems 2.4–5. The proofs are contained in Sect. 3–5.

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