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Conservation Laws and Symmetries of Generalized Sine-Gordon Equations

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Abstract. We study some systems of non-linear PDE's (Eqs. 1.1 below) which can be regarded either as generalizations of the sine-Gordon equation or as two-dimensional versions of the Toda lattice equations. We show that these systems have an infinite number of non-trivial conservation laws and an infinite number of symmetries. The second result is deduced from the first by a variant of the Hamiltonian formalism for evolution equations. We also consider some specializations of the systems.

1. Introduction

The title refers to the following system of equations for n unknown functions $R_0(x,t),\ldots,R_{n-1}(x,t)$:

$$R_{i,xt} = c_{i-1}\exp(R_{i-1}-R_i) - c_i\exp(R_i - R_{i+1}). \tag{1.1}$$

The c_i are constants, and the suffixes are read mod n where necessary. It follows from (1.1) that $(\Sigma R_i)_{xt} = 0$, and the most interesting case is when $\Sigma R_i = 0$ too, so that there are really only $n-1$ independent unknown functions in (1.1), say $R_0,\ldots,R_{n-2}$. However, even in this case it is more pleasant to write the equations in the symmetrical form (1.1).

These equations have been studied recently by several other authors (see [4, 9, 10]). The work [10] is in some respects more advanced than ours: we did not see either this paper or [9] until the present manuscript had been completed. We feel that since our point of view is rather different from that of [10], it is best to present our results without any alteration. However, at the end of the introduction we have inserted a few comments comparing our results with those of [10].

Let us first explain how the Eqs. (1.1) arise from our point of view. In the simplest case $n=2$ and $R_0 + R_1 = 0$, we have just one unknown $R = R_0$, and the equation is

$$R_{xt} = c_1\exp(-2R) - c_0\exp(2R). \tag{1.2}$$

For suitable values of c_i we get the well known sinh-Gordon equation

$$R_{xt} = \sinh 2R.$$

The factor 2 is of course inessential, and could be removed by rescaling. Replacing

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