Commun. Math. Phys. 81, 39–88 (1981)

Communications in Mathematical Physics

Absolutely Continuous Invariant Measures for One-Parameter Families of One-Dimensional Maps

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Abstract. Given a one-parameter family $f_\lambda(x)$ of maps of the interval [0, 1], we consider the set of parameter values λ for which f_λ has an invariant measure absolutely continuous with respect to Lebesgue measure. We show that this set has positive measure, for two classes of maps: i) $f_\lambda(x)=\lambda f(x)$ where $0<\lambda\leqq 4$ and $f(x)$ is a function C^3-near the quadratic map $x(1-x)$, and ii) $f_\lambda(x)=\lambda f(x)$ (mod 1) where f is C^3, $f(0)=f(1)=0$ and f has a unique nondegenerate critical point in [0, 1].

0. Introduction

Dynamical systems generated by noninvertible maps of an interval into itself have been intensely studied recently. The most widely considered was the family $f_\lambda: x\to\lambda x(1-x)$, $x\in[0,1]$, $0\leqq\lambda\leqq 4$.

It is well-known that if f_λ has an attracting periodic orbit $\bar{\alpha}=(\alpha_1,\ldots,\alpha_n)$ then all probabilisitic f_λ-invariant measures are singular with respect to a Lebesgue measure dx, and the iterations $f^n_{\lambda *}dx$ converge in the weak *-topology to the discrete invariant measure supported by $\bar{\alpha}$.

It is probable (but not proved) that this situation is typical from the topological point of view, i.e. for a general one-parameter family of smooth mappings $f_\lambda: I\to I$, $\lambda\in\Lambda$, there is an open and dense subset Λ_0 of Λ such that for $\lambda\in\Lambda_0$, the set of limit points for $f^n_{\lambda *}dx$ consists of a finite number of measures supported by periodic attracting orbits.

We show in the present paper that this is not so from the metric point of view. Namely we prove for a certain class of one-parameter families f_λ that the set

$$\Lambda_1=\{\lambda: f_\lambda \text{ has an invariant finite measure } \mu_\lambda \text{ absolutely continuous with respect to } dx\ (\mu_\lambda<dx)\}$$

has a positive measure in Λ.

In the classical case $x\to 4x(1-x)$ considered by Ulam and von Neumann in [1], the invariant measure $\mu(dx)$ has density $\varrho(x)=\dfrac{1}{\pi\sqrt{x(1-x)}}$. In [2] Bunimovič

0010-3616/81/0081/0039/$10.00