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On the Bäcklund Transformation for the Gel'fand-Dickey Equations

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Abstract. We study the Bäcklund transformation of the Gel'fand-Dickey equations, and in particular how the factorization of n^{th} order differential operators leads to Lax type equations for first order operators, generalizing work of Adler and Moser [1]. In a similar fashion we study the Toda equations.

1. Introduction

In the study of the Korteweg-deVries (KdV) equation,

$$q_t = 3qq_x - \tfrac{1}{2}q_{xxx}, \tag{1.1}$$

a Bäcklund transformation for (1.1) can be made to play an important role, as in [1]. As is well known, (1.1) can be rewritten in the Lax form

$$\begin{gathered} \frac{dL}{dt} = [P, L], \qquad L = L(q) = -D^2 + q, \\ P = P(q) = D^3 - \tfrac{3}{4}(qD + Dq), \qquad D = \frac{\partial}{\partial x}. \end{gathered} \tag{1.2}$$

Factoring

$$L = A^T A, \qquad A = D - v, \qquad A^T = -D - v,$$

we find $q = q(v) = v_x + v^2$, and the Bäcklund transformation for (1.1), $q(v) \mapsto q(-v)$, corresponds to reordering the factors of L, i.e.,

$$L = A^T A \mapsto \tilde{L} \equiv AA^T, \qquad q(v) \mapsto q(-v). \tag{1.3}$$

The crucial point is that the transformation $q(v) \mapsto q(-v)$ preserves (1.1). The best way to see that for our purposes is to observe that if v satisfies the so-called

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