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Ergodic Properties of Equilibrium States

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Abstract. Possible ergodic properties of Gibbs states are discussed by constructing a number of examples. In particular existence of Gibbs states which are mixing but not extremal is shown.

Any invariant state ϱ which is an extremal Gibbs state, [2], [6], [8], is mixing [6], i.e. for any $f, g \in L^2(\varrho)$

$$\varrho(f \cdot \tau_a g) \to \varrho(f)\varrho(g) \quad \text{as} \quad |a| \to \infty .$$

In this note we discuss the question to what extent the converse holds, i.e. if any mixing Gibbs state is extremal. In particular, we construct an example of a mixing non-extremal Gibbs state.

More precisely, we consider a hierarchy of ergodic properties: ergodicity, weak mixing, n-fold mixing ($n > 2$), extremality in the set of all Gibbs states, each property being stronger than the preceding one. By considering suitable ferromagnetic finite range interactions we obtain equilibrium states which are

– ergodic but not weakly mixing;

– weakly mixing but not mixing;

– mixing (i.e., 2-fold mixing) but not 3-fold mixing, and thus not extremal Gibbs states.

Construction of ergodic but not weakly mixing states is trivial: average over translations of any periodic not $\mathbb{Z}^\nu$-invariant extremal Gibbs state yields here an example. One can obtain such examples already in two dimensions. The other examples are more complicated and are constructed in dimension three or higher. At the end of this note we discuss a result of Ledrappier [7] which stimulated working out of the examples below.

We adopt the following definitions [4], [1]. Let $\mathscr{X}$ be a compact $\mathbb{Z}^\nu$-space (i.e., $\mathbb{Z}^\nu$ acts on $\mathscr{X}$ by homeomorphisms). We denote by $f \mapsto \tau_a f$, $a \in \mathbb{Z}^\nu$, the induced action of $\mathbb{Z}^\nu$ on $C(\mathscr{X})$ and by E^I the family of all $\mathbb{Z}^\nu$-invariant states of $C(\mathscr{X})$. For

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