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Geometrical Structure and Ultraviolet Finiteness in the Supersymmetric σ-Model

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Abstract. A complete geometrical classification of supersymmetric σ-models is given. Extended supersymmetry requires covariantly constant complex structures, and Kahler and hyperkahler manifolds play a special role. As an application of the classification, it is shown that a particular class of these models is on-shell ultraviolet finite to all orders in perturbation theory.

Nonlinear σ-models are the quantum field theories of harmonic maps from space-time into a Riemannian manifold M. Recent work indicates that there is an intimate connection between supersymmetric versions of the models and differential geometry. There is a correlation between complex manifolds and extended supersymmetry, [1, 2] with the strong implication for models in two space-time dimensions that ultraviolet divergences are severely limited compared to expectations based on power counting [3]. In particular when the manifold M is Ricci-flat the associated supersymmetric σ-model is on-shell ultraviolet finite through 3-loop order [3–5]. For general manifolds there appears to be at most a 1-loop contribution [3] to the generalized β-function [6].

In this paper we present new results of two types.

I. A complete geometrical characterization of manifolds which permit extended supersymmetry is given. The possibilities are $N=2$ supersymmetry which occurs if and only if M is Hermitian and Kahler [7] and $N=4$ supersymmetry which requires that M is hyperkahler [8].

II. Strong restrictions on the on-shell ultraviolet counterterms are derived by combining general considerations of power counting and invariance with the complex manifold structure required by extended supersymmetry. In particular it is shown that the allowed ultraviolet counterterms of $N=4$ models must be zero modes of the Lichnerowicz Laplacian on M which are also algebraic functions of the curvature tensor. For a subclass of $N=4$ models, namely those for which M is a four-dimensional asymptotically locally Euclidean self-dual gravitational instanton [9], it is shown that there are no solutions to these requirements. There are therefore no on-shell ultraviolet counterterms to any order in perturbation theory.

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