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Conserved Densities for Linear Evolution Systems

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Abstract. A general discussion of the conservation laws for simple linear evolution systems is presented. The analysis is based upon an extension of the Gel'fand–Dikii symbolic algorithm to cover pseudo-differential operators. These techniques are applied to obtain all the conserved densities $\varrho[u]$ for the free Klein–Gordon and Dirac equations with nonzero mass.

1. Introduction

In previous papers [1–3] we have discussed the polynomial conserved densities for a single (linear or nonlinear) evolution equation. However, many other relevant equations in mathematical physics such as the Klein–Gordon equation, wave equation, and more generally, evolution systems like Dirac's were not covered by such analysis.

The aim of this paper is to present a full treatment of the conservation laws for simple (i.e. diagonalizable) linear evolution systems which greatly generalize older results [1]. This is accomplished by enlarging through Fourier techniques the Gel'fand–Dikii symbolic calculus [4] and allows us to use pseudo-differential operators. As an important byproduct, it is shown that all conserved densities for simple linear evolution systems are at most quadratic in the field variables barring the exceptional cases where one or more of the diagonalized evolution equations is of the form $v_t = (\mathbf{a} \cdot \mathbf{D} + b)v$, $\mathbf{a}$, b constant. In particular, there it follows that for the free Klein–Gordon equation [5] and Dirac system with nonzero mass, any conserved density is quadratic in the fields.

In Sect. 2 we briefly expose the Fourier–Gel'fand calculus. Section 3 contains a detailed discussion of the conserved densities for simple linear evolution systems and Sect. 4 applies these results to linear evolution equations, to wave-like equations and to the Dirac system.

Appendix A contains two illustrative examples of some peculiar situations.

Finally, Appendix B establishes the relationship between linear evolution systems and individual partial differential equations (PDE's) for each field

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