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# The Field Copy Problem: to what Extent do Curvature (Gauge Field) and its Covariant Derivatives Determine Connection (Gauge Potential)?

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**Abstract.** We show that a connection of a principal bundle is determined up to (global) gauge equivalence by the curvature and its covariant derivatives provided that the infinitesimal holonomy group is of constant dimension and the base space is simply connected. If the dimension of the infinitesimal holonomy group varies, there may be obstructions of a topological nature to the existence of a global or even local gauge equivalence between two connections whose curvatures and covariant derivatives of curvature agree everywhere. These obstructions are analyzed and illustrated by examples.

## 1. Introduction and Notation

We follow the definitions and conventions of Kobayashi and Nomizu [7]. Let $P$ be a fixed principal bundle with gauge group $G$ over a base space $M$, and let $\pi: P \to M$ be the bundle projection. The group $G$ may be any Lie group (we denote its Lie algebra by $\mathfrak{g}$), while $M$ and $P$ are smooth ($=C^\infty$) manifolds, and $\pi$ is a smooth map. We denote the (right) action of $G$ on $P$ by $(u,g) \mapsto u \cdot g \in P$, where $u \in P$ and $g \in G$. Let $\omega$ be a ($\mathfrak{g}$-valued) connection 1-form on $P$ and $\Omega$ its curvature 2-form, as defined in [7]. In physical terms, $\omega$ is a gauge potential (usually denoted $A$ in physics), and $\Omega$ (denoted $F$ in physics) is the gauge field it determines. If $X$ is any smooth vector field on $M$, let $\tilde{X}$ denote the ($\omega$-)horizontal lift of $X$ to $P$. [Thus $\omega(\tilde{X})=0$ and $\pi_* \tilde{X}_u = X_{\pi(u)}$ for all $u \in P$.] If $f$ is a $C^\infty$ vector-valued function on $P$, one defines the *covariant derivative of $f$ along $X$* to equal the usual derivative $\tilde{X}f$ of $f$ along $\tilde{X}$.

Given local coordinates $(x_1, \ldots, x_n)$ on an open set $V \subset M$, we let $\partial_i = \partial/\partial x_i$ and denote $\tilde{\partial}_i$ by $D_i$, so that $D_i f$ is the covariant derivative of $f$ in the $i$th direction. Now choose any $C^\infty$ local trivialization $P|V \xrightarrow{\approx} V \times G$. This is equivalent to choosing a local gauge (section) on $V$, namely, the section corresponding to $V \times \{1\} \subset V \times G$, where 1 is the identity element of $G$. Then the restriction $\omega | V \times G$ equals $\theta + \sum_{i=1}^{n} A_i dx_i$, where $\theta$ is the canonical left-invariant $\mathfrak{g}$-valued 1-form on $G$ and the $A_i : V \times G \to \mathfrak{g}$ are smooth functions of *type ad*, that is, $A_i(u \cdot g)$

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