Commun. Math. Phys. 78, 19–63 (1980)

Communications in Mathematical Physics

A Limit Theorem for Stochastic Acceleration

H. Kesten[1] and G. C. Papanicolaou[2]

[1]Department of Mathematics, Cornell University, Ithaca, NY 14853, USA

[2]Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA

Abstract. We consider the motion of a particle in a weak mean zero random force field F, which depends on the position, $x(t)$, and the velocity, $v(t) = \dot{x}(t)$. The equation of motion is $\ddot{x}(t) = \varepsilon F(x(t), v(t), \omega)$, where $x(\cdot)$ and $v(\cdot)$ take values in $\mathbb{R}^d$, $d \geqq 3$, and ω ranges over some probability space. We show, under suitable mixing and moment conditions on F, that as $\varepsilon \to 0$, $v^\varepsilon(t) \equiv v(t/\varepsilon^2)$ converges weakly to a diffusion Markov process $v(t)$, and $\varepsilon^2 x^\varepsilon(t)$ converges weakly to $\int_0^t v(s)ds + x$, where $x = \lim \varepsilon^2 x^\varepsilon(0)$.

1. Introduction

For simplicity we do not discuss the general situation in this section, but restrict ourselves to force fields which depend on position only.

Let $F(x), x \in \mathbb{R}^d$, be a random vector field, a random force field, which is stationary and has mean zero. Let $x(t)$ be the coordinate of a particle of unit mass moving through this force field. The equation of motion is

$$\ddot{x} = F(x). \tag{1.1}$$

with given initial position and velocity. Suppose that the force is weak and weakly correlated for points that are far apart. Then one expects that after a long time the velocity $\dot{x}$ will behave like a diffusion Markov process and the position x like the integral of this diffusion process.

To be more specific, suppose that the root mean square of the force field F is proportional to ε so that we may replace (1.1) by

$$\ddot{x} = \varepsilon F(x) \tag{1.2}$$

in which $F(x)$ is of order one. Rescaling of time t into t/ε^2 and putting $\dot{x}(t/\varepsilon^2) = v^\varepsilon(t)$, $x(t/\varepsilon^2) = x^\varepsilon(t)$ leads from (1.1) to the system

$$\frac{dx^\varepsilon(t)}{dt} = \frac{1}{\varepsilon^2} v^\varepsilon(t)$$

$$\frac{dv^\varepsilon(t)}{dt} = \frac{1}{\varepsilon} F(x^\varepsilon(t)) \tag{1.3}$$

0010-3616/80/0078/0019/$09.00