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Self-Dual Space-Times with Cosmological Constant

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Abstract. It is shown that self-dual solutions of Einstein's equations, with cosmological constant λ, correspond to certain complex manifolds. This result generalizes the work of Penrose [1], who dealt with the case $\lambda = 0$.

1. Introduction

Recently there has been much interest in four-dimensional spaces which have self-dual conformal curvature, and which satisfy Einstein's vacuum equations (with or without cosmological constant) [1–5]. This is despite the fact that such spaces cannot have Lorentzian signature, and therefore have no direct application to classical gravity. The work in the papers listed above seems to suggest that self-dual solutions underlie gravity; that they are (in some sense) its basic constituents. Be that as it may, there is no doubt that self-dual solutions are a good deal easier to characterize and construct then more general solutions. This paper presents a characterization of self-dual solutions in terms of complex manifolds; it is essentially a generalization of Penrose's 'Nonlinear Gravitons and Curved Twistor Theory' [1], which deals only with the case of zero cosmological constant. Penrose's result has proved to be very useful for constructing and understanding self-dual solutions of Einstein's equations [6, 7]. In addition, solutions of the wave equation $(\Box + \frac{1}{6}R)\phi = 0$ and of massless field equations of higher spin, in a self-dual gravitational background, may be neatly characterized in terms of certain sheaf cohomology groups [8]. Using this, one can explicitly construct Green's functions for these equations in a very natural way [9].

The remainder of this section is devoted to setting up some notation and conventions. Then in Sect. 2 the main theorem is presented, establishing a one-to-one correspondence between self-dual solutions of Einstein's equations and deformations of flat twistor space which preserve certain differential forms. Some details of the construction and proof are relegated to Sect. 3 and an Appendix. In Sect. 4 the CP_2 gravitational instanton [10, 11] is discussed by way of example.

The following index conventions are used: $a, b, \ldots$ are 4-dimensional space-time indices, $A, B, \ldots, A', B', \ldots$ are 2-dimensional spinor indices, and $\alpha, \beta, \ldots$ are 4-dimensional twistor indices. The tangent space T at a point of space-time is isomorphic to the tensor product of the unprimed spin-space S and the primed

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