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A Commutativity Theorem of Partial Differential Operators

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Abstract. Let $u=u(x,t)$ be a function of x and t, and $u_i=\mathscr{D}^i u$, $\mathscr{D}=d/dx$, $i=0,1,2,\ldots$, be its derivatives with respect to x. Denote by $\mathbf{W}_n$ the set $\{f|f=f(u,u_1,\ldots,u_n),(\partial/\partial u_n)f\neq 0\}$, where $f(u,\ldots,u_n)$ are polynomials of u_i with constant coefficients. To any $f\in\mathbf{W}=\bigcup_{n=2}^{\infty}\mathbf{W}_n$, we relate it with an operator $\mathscr{U}(f)=\sum_{i\geqq 0}(\mathscr{D}^i f)\partial/\partial u_i$. In this paper we prove that: $\mathscr{U}(f)$ commutes with $\mathscr{U}(g)$ if they commute respectively with $\mathscr{U}(h)$, provided f, g, $h\in\mathbf{W}$. Relating to this commutativity theorem, we prove that, if an evolution equation $u_t=f(u,\ldots,u_n)$ possesses nontrivial symmetries (or conservation laws for a class of polynomials f), then $f=Cu_n+f_1(u,\ldots,u_r)$, where $C=\text{const}$, and $r<n$. In the end of this paper, we state a related open problem whose solution would be of much value to the theory of soliton.

1. Introduction

The soliton [1], being a particle-like solution of the nonlinear wave equation, has been now applied widely in various fields of physics. In the recent years, a number of interesting mathematical problems have arisen in the study of soliton, one of them is, among other things, the commutativity of differential operators [2, 3]. Let $\mathscr{A}=a_0\mathscr{D}^n+\ldots+a_{n-1}\mathscr{D}+a_n$, $a_0\neq 0$, $\mathscr{D}=d/dx$ be a differential operator, and $\mathbf{C}(\mathscr{A})$ be the set of all linear operators which commute with $\mathscr{A}$. A pronounced results [4] is the fact that $\mathbf{C}(\mathscr{A})$ is a commutative ring. In this paper we established a similar result concerning the partial differential operators $\mathscr{U}(f)=\sum_{i\geqq 0}(\mathscr{D}^i f)\partial/\partial u_i$, where $f=f(u,\ldots,u_n)$ are polynomials of $u_i=\mathscr{D}^i u$ with constant coefficients, and $u=u(x,t)$ is a sufficiently smooth function of x and t. We discuss further the application of this result to the study of symmetries and conservation laws of nonlinear evolution equations.

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