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Stochastic Coupling and Thermodynamic Inequalities*

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Abstract Let μ_1 and μ_2 be thermodynamic Gibbs measures on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Diffusions are constructed having μ_1, and μ_2 as invariant measures. These diffusions are then coupled; inequalities between expectations of certain random variables on the two spaces result.

1. Introduction

Let (X, μ_1) and (Y, μ_2) be two probability measure spaces and let $\phi_1, \ldots, \phi_k$ and $\psi_1, \ldots, \psi_k$ be families of random variables on X and Y respectively. Then we say μ_1 is stochastically less than μ_2 with respect to $\phi_1, \ldots, \phi_k; \psi_1, \ldots, \psi_k$, and write $\mu_1 \overset{s}{\leqq} \mu_2$, if for all $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$, $\mu_1\{\phi_1 \geqq \lambda_1, \ldots, \phi_k \geqq \lambda_k\} \leqq \mu_2\{\psi_1 \geqq \lambda_1, \ldots, \psi_k \geqq \lambda_k\}$. Note that if $\mu_1 \overset{s}{\leqq} \mu_2$ and f is an increasing function on $\mathbb{R}$, $\int f(\phi_j) d\mu_1 \leqq \int f(\psi_j) d\mu_2$, from integration by parts; stochastic inequality of measures allows one to compare expectations of certain random variables on X and Y. The purpose of this article is to describe conditions under which this stochastic comparison can be made, in the setting where μ_1 and μ_2 are thermodynamic Gibbs measures on $\mathbb{R}^m$ and $\mathbb{R}^n$ and $\phi_1, \ldots, \phi_k$ and $\psi_1, \ldots, \psi_k$ are linear functions on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

The basic ideas of the paper are illustrated in the simple example of $d\mu_1(x) = \exp(-H_1(x))dx$, $d\mu_2(y) = \exp(-H_2(y))dy$ both probability measures on $\mathbb{R}$, $\phi(x) = x$, $\psi(y) = y$, and H_1 and H_2 smooth. Let $K = \frac{1}{2}(\nabla_x + \nabla_y)^2 - \frac{1}{2}(\nabla_x H_1)\nabla_x - \frac{1}{2}(\nabla_y H_2)\nabla_y$ be a differential operator acting on continuous functions on $\mathbb{R}^2$. (From an operator standpoint, the "coupling" is the cross term $\nabla_x \nabla_y$ in K.) Under suitable conditions, K is the generator of a semigroup $\exp(tk)$ representable by $\exp(tK) f(x_0, y_0) = Ef(x(t, x_0), y(t, y_0))$ with E expectation with respect to Brownian motion and $(x(t, x_0), y(t, y_0))$ the solution to a *coupled* set of stochastic differential equations. Suppose $\nabla_x H_1(x) \geqq \nabla_x H_2(x)$. Then these equations yield a stochastic differential inequality which implies that $x(t, x_0) \leqq y(t, y_0)$ if the initial values for $x(t, x_0)$, $y(t, y_0)$ satisfy $x_0 \leqq y_0$. Let $\hat{\mu}$ be a probability measure on $\mathbb{R}^2$ supported in the region $x \leqq y$ and let $g_1(x, y) = f(x)$, $g_2(x, y) = f(y)$ with f a continuous

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