Commun. Math. Phys. 77, 149–172 (1980)

Communications in Mathematical Physics

Nonlinear Desingularization in Certain Free-Boundary Problems*

M. S. Berger[1] and L. E. Fraenkel[2]

[1] Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003, USA, and
[2] Mathematics Division, University of Sussex, Brighton BN1 9QH, England

Abstract. We consider a nonlinear, elliptic, free-boundary problem involving an initially unknown set A that represents, for example, the cross-section of a steady vortex ring or of a confined plasma in equilibrium. The solutions are characterized by a variational principle which allows us to describe their behaviour under a limiting process such that the diameter of A tends to zero, while the solutions degenerate to the solution of a related linear problem. This limiting solution is the sum of the Green function of the linear operator and of a smooth function satisfying the boundary conditions. Mathematically speaking, this limiting process, that we call "nonlinear desingularization", is a novel kind of bifurcation phenomenon since the nonlinear effect here involves smoothing the singularity of the associated linear problem.

"Nonlinear desingularization" is an interesting, hitherto little studied phenomenon in nonlinear partial differential equations. By nonlinear desingularization we mean that a linear boundary-value problem, whose solution possesses one or more isolated singularities, is a degenerate form of a family of nonlinear problems whose solutions are smooth and, moreover, converge to the solution of the linear system upon degeneration. This phenomenon occurs, for example, in the study of vortex motion in ideal fluids; there a circular vortex filament is used to approximate a steady vortex ring of small cross-section. The Stokes stream function of the filament is the Green function of an elliptic operator, and hence has an isolated singularity, while the Stokes stream function for the vortex ring of small cross-section satisfies a non-linear partial differential equation, and is smooth. In this paper we study this situation from a global point of view that is independent of the elaborate singular-perturbation techniques often used for such problems.

Nonlinear desingularization arises in problems of theoretical physics other than the classical problem mentioned above. For example, it occurs in the onset of flux penetration, as determined by the Ginzberg-Landau theory, in supercon-

* Research partially supported by AFOSR and NSF grants

0010-3616/80/0077/0149/$04.80