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Lieb's Correlation Inequality for Plane Rotors

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Abstract. We prove a conjecture by E. Lieb, which leads to the Lieb inequality for plane rotors. As in the Ising model case, this inequality implies the existence of an algorithm to compute the transition temperature of this model.

Introduction

In [2], Simon proved and applied certain correlation inequalities. These inequalities are special cases of a class of inequalities proved earlier by Boel and Kasteleyn [3–6]. For a finite range pairwise interacting Ising ferromagnet on a lattice L, the inequalities in [2] are:

$$\langle s_a \cdot s_c \rangle < \sum_{b \in B} \langle s_a \cdot s_b \rangle \langle s_b \cdot s_c \rangle , \tag{1}$$

where B is any set of spins separating a from c.

Lieb has generalized this inequality. In [1], he showed the stronger assertion:

$$\langle s_a \cdot s_c \rangle \leqq \sum_{b \in B} \langle s_a \cdot s_b \rangle_A \langle s_b \cdot s_c \rangle , \tag{2}$$

where A is the union of B and the connected component of L–B containing a, and $\langle \cdot \rangle_A$ denotes expection values with respect to the A system only. He also reduced the proof of a similar inequality for plane rotors to a conjecture on directed graphs, which he proves in a special case, and which we prove generally in the next section. Hence we obtain:

Theorem. *Let us consider a plane rotors model with pairwise interaction between two spins* $\mathbf{s}_a$ *and* $\mathbf{s}_b$ *of type* $-J_{a,b}\mathbf{s}_a \cdot \mathbf{s}_b$, *with* $J_{a,b} \geqq 0$.
Then

$$\langle \mathbf{s}_a \cdot \mathbf{s}_c \rangle \leqq \sum_{b \in B} \langle \mathbf{s}_a \cdot \mathbf{s}_b \rangle_A \langle \mathbf{s}_b \cdot \mathbf{s}_c \rangle , \tag{3}$$

where B separates a from c, and A has same meaning as previously.

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