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A Refinement of Simon's Correlation Inequality*

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Abstract. A general formulation is given of Simon's Ising model inequality: $\langle\sigma_\alpha\sigma_\gamma\rangle \leqq \sum_{b\in B}\langle\sigma_\alpha\sigma_b\rangle\langle\sigma_b\sigma_\gamma\rangle$ where B is any set of spins separating α from γ. We show that $\langle\sigma_\alpha\sigma_b\rangle$ can be replaced by $\langle\sigma_\alpha\sigma_b\rangle_A$ where A is the spin system "inside" B containing α. An advantage of this is that a finite algorithm can be given to compute the transition temperature to any desired accuracy. The analogous inequality for plane rotors is shown to hold if a certain conjecture can be proved. This conjecture is indeed verified in the simplest case, and leads to an upper bound on the critical temperature. (The conjecture has been proved in general by Rivasseau. See notes added in proof.)

In an accompanying paper [1] in this volume Simon proves a correlation inequality with important consequences. For a finite range pairwise interacting (generalized) Ising ferromagnet (the spins take on values $2M, 2M-2, \ldots, -2M$), Simon shows that

$$\langle\sigma_\alpha\sigma_\gamma\rangle \leqq \sum_{b\in B}\langle\sigma_\alpha\sigma_b\rangle\langle\sigma_b\sigma_\gamma\rangle\,, \tag{1}$$

where B is any set of spins separating α from γ (i.e. any path from α to γ must run through B). Aizenman and Simon [2] have proved a related inequality for N-component spins. In this paper we shall generalize (1) in the following way: $\langle\sigma_\alpha\sigma_b\rangle$ can be replaced by $\langle\sigma_\alpha\sigma_b\rangle_A$, where A is the connected component of the lattice containing α and B and $\langle\cdot\rangle_A$ denotes expectation values in the A system alone. The possibility of extending this inequality to plane rotors is also discussed, but the proof is carried to completion only in a special case. (See notes added in proof.)

In [1] Simon discusses the consequences of (1) and our generalization. We shall not repeat them, except to note that the most interesting consequence of the extension is that for the first time one has an algorithm for computing the transition temperature, T_c (in the sense that above, but not below T_c there is

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