Commun. Math. Phys. 77, 21–30 (1980)

Communications in
Mathematical
Physics

Integrable Nonlinear Klein-Gordon Equations and Toda Lattices

Allan P. Fordy and John Gibbons

School of Theoretical Physics, Dublin Institute for Advanced Studies,
Dublin 4, Ireland

Abstract. We present a class of nonlinear Klein-Gordon systems which are soluble by means of a scattering transform. More specifically, for each $N \geqq 2$ we present a system of $(N-1)$ nonlinear Klein-Gordon equations, together with the corresponding $N \times N$ matrix scattering problem which can be used to solve it. We illustrate these with some special examples. The general system is shown to be closely related to the equations of the periodic Toda lattice. We present a Bäcklund transformation and superposition formula for the general system.

Introduction

Part of the current folk-lore in soliton theory is that the only integrable, nonlinear Klein-Gordon equations in one variable θ:

$$\theta_{xt} = F(\theta) \tag{1.1}$$

are those for which

$$\frac{d^2F}{d\theta^2} + kF = 0. \tag{1.2}$$

This belief stems from two distinct lines of argument; one is the study of self-Bäcklund transformations, and the other is the question of the existence of an infinite number of polynomial conserved densities (p.c.d.s).

A well known result is that Eq. (1.1) possesses a self-Bäcklund transformation if and only if condition (1.2) is satisfied [1–3].

The question of the existence of p.c.d.s is more complicated, however. The sufficiency of (1.2) for the existence of an infinite number of p.c.d.s is well known, but the necessity has never been proven. Kruskal [4] looked for a conserved density with leading order term θ_{xx}^2 (of weight 4) for Eq. (1.1), and found (1.2) to be necessary for its existence. Dodd and Bullough [5] later found several higher order p.c.d.s for the equation:

$$\theta_{xt} = e^{2\theta} - e^{-\theta} \tag{1.3}$$

but believed, at the time, that only a finite number of those existed.

0010-3616/80/0077/0021/$02.00