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Riemannian Structure on Manifolds of Quantum States

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Abstract. A metric tensor is defined from the underlying Hilbert space structure for any submanifold of quantum states. The case where the manifold is generated by the action of a Lie group on a fixed state vector (generalized coherent states manifold hereafter noted G.C.S.M.) is studied in details; the geometrical properties of some wellknown G.C.S.M. are reviewed and an explicit expression for the scalar Riemannian curvature is given in the general case. The physical meaning of such Riemannian structures (which have been recently introduced to describe collective manifolds in nuclear physics) is discussed. It is shown on examples that the distance between nearby states is related to quantum fluctuations; in the particular case of the harmonic oscillator group the condition of zero curvature appears to be identical to that of non dispersion of wave packets.

1. Introduction

In recent years there has been renewed interest in the usefulness of geometrical ideas in quantum mechanics. The geometrical structure which has been most studied is the symplectic one. The reason is that the symplectic forms take an important part in the Hamiltonian formulation of the classical mechanics [1] and that they can also be defined on the Hilbert space of quantum states. The key role of this structure is particularly evident in the geometrical quantization program of Kostant and Souriau [2]. It has also been claimed that this geometrical structure *remains present in the nonlinear generalizations of the quantum mechanics [3].*

Another useful geometrical concept is that of a Riemannian structure. In the framework of quantum mechanics this notion has not been much investigated. An explanation may be that, although the scalar product on the Hilbert space induces naturally a distance between the quantum states, one is not interested in the local properties of the manifold of states. Indeed the physically relevant quantities are

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