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Universal Properties of Maps on an Interval

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Abstract. We consider iterates of maps of an interval to itself and their stable periodic orbits. When these maps depend on a parameter, one can observe period doubling bifurcations as the parameter is varied. We investigate rigorously those aspects of these bifurcations which are universal, i.e. independent of the choice of a particular one-parameter family. We point out that this universality extends to many other situations such as certain chaotic regimes. We describe the ergodic properties of the maps for which the parameter value equals the limit of the bifurcation points.

1. Introduction

Continuous mappings of intervals into themselves display some remarkable properties when regarded as discrete dynamical systems. (For a survey, see May [9] or Collet and Eckmann [14].) One much-studied example is the one-parameter family

$$x \to 1 - \mu x^2 \tag{1.1}$$

which maps $[-1,1]$ into itself for $0 \leqq \mu \leqq 2$. In this and similar examples, what is interesting is not so much the behavior of any particular mapping; rather, it is the way this behavior changes with μ.

The example (1.1), and the more general one-parameter families $\mu \to \psi_\mu$ we will study, have a simplifying qualitative feature: Each ψ_μ has a unique (differentiable) maximum – at $x=0$ in the example – below which it is increasing and above which it is decreasing. We will consider mappings ψ which satisfy

P1) ψ is a continuously differentiable mapping of $[-1,1]$ into itself.

P2) $\psi(0)=1$; ψ is strictly increasing on $[-1,0]$ and strictly decreasing on $[0,1]$.

P3) $\psi(-x)=\psi(x)$.

The space of all such mappings will be denoted by $\mathscr{P}$. (The condition that the maximum of ψ occurs at zero and that ψ sends zero to one can frequently be arranged, if necessary, by making an affine change of variables.) We have included

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