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An Inequality for Trace Ideals

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Abstract. We prove an inequality for trace ideals which relates the difference of two positive operators to the difference of their square roots. Inequalities involving operator-monotone functions more general than the square root, are considered as well.

1. Introduction

In practical work involving, for instance, the approach to an equilibrium in a harmonic chain [1], or the elementary excitation spectrum of a random ferromagnet [2], the following problem shows up. One is given two positive bounded operators whose difference is finite-dimensional, and one would like to prove that the difference of the square roots is trace class [3]. So we are led to the question: If A and B are positive bounded operators such that $(A-B)$ is trace class, when is it true that the difference of their square roots is trace class as well? We cannot expect that this is true in general. Simply take $B=0$ and A trace class; $A^{1/2}$ is Hilbert-Schmidt, but not necessarily trace class. So we need a supplementary condition. In Sect. 2 (Proposition 2.1) we will meet such a condition and get an estimate of the trace norm $\|A^{1/2}-B^{1/2}\|_1$ in terms of $\|A-B\|_1$. In connection with this estimate we then may look for meaningful generalizations.

They are two natural ways of generalizing the above problem. We consider them in turn. First we notice that the function $f(\lambda)=\lambda^{1/2}$ is of a very special nature. It has the property that for any positive A and B such that $A \leqq B$ we have $f(A) \leqq f(B)$. Such a function is called operator-monotone [4]. Is it true that Proposition 2.1, when appropriately modified, also holds for operator-monotone functions?

Next we observe that up to now we have singled out the trace norm. But what can be said if we replace the trace norm, which is in fact an l^1-norm, by an l^p-norm or, even more generally, by a symmetric norm [5, 6]? This question is considered in Sect. 3. Operator-monotone functions are included in the analysis in Sect. 4.

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