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Existence of a Homoclinic Point for the Hénon Map

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Abstract. We prove analytically that for the Hénon map of the plane into itself $(s, t) \mapsto (t+1-1.4a^2, 0.3s)$, there exists a transversal homoclinic point.

Curry in his paper [1] investigates numerically the map $T: \mathbb{R}^2 \to \mathbb{R}^2$:

$$T(s,t) = (t+1-as^2, bs), \qquad a=1.4, \qquad b=0.3$$

(defined first by Hénon [2]). His results suggest the existence of a transversal homoclinic point. However, he writes that his arguments cannot be considered as a rigorous proof. Moreover, he is sceptical about the possibility of obtaining such a proof with a computer.

For some values of a and b, the existence of a transversal homoclinic point was proved analytically by Marotto [3], but it seems impossible that his methods could be applied to the case $a=1.4$, $b=0.3$.

In the present paper we show that the problem (for $a=1.4$, $b=0.3$) is less complicated than it looks. We obtain an analytical proof just by making appropriate estimates. The most complicated computations are of the type $1.42 \times 0.626 = 0.88892$, therefore a reader who wants to check all of them needs only a pen and paper (although a pocket calculator is of course better). We omit some computations in Step 1, but they are standard, and besides, they were already done by Hénon.

We introduce the new coordinates: $x = \frac{t}{0.3}$, $y = s$. In these coordinates the map has the form

$$f(x,y) = (y, 1-1.4y^2+0.3x).$$

It has one (minor) advantage: when we take an image, or an inverse image of a point then we introduce only one new number.

The map f has a hyperbolic fixed point $P = (x_0, x_0)$, where

$$x_0 = \frac{-0.7 + \sqrt{6.09}}{2.8}.$$

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