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On the Equivalence of the First and Second Order Equations for Gauge Theories*

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Abstract. We prove that every solution to the SU(2) Yang–Mills equations, invariant under the lifting to the principle bundle of the action of the group, O(3), of rotations about a fixed line in $\mathbb{R}^4$, with locally bounded and globally square integrable curvature is either self-dual or anti-self dual. In other words we prove, under the above assumptions, that every critical point of the Yang–Mills functional is a global minimum.

We prove also that every finite extremal of the Ginzburg–Landau action functional on $\mathbb{R}^2$, with the coupling constant equal to one, is a solution to the first order Ginzburg–Landau equations. The relationship between the Ginzburg–Landau equations and the O(3) symmetric, SU(2) Yang–Mills equations on $\mathbb{R}^2 \times S^2$ is established.

I. Introduction

On Euclidean four space the value of the Yang–Mills action evaluated on a connection is bounded below by the topological invariant. Any connection whose action achieves this minimum is a solution to the Yang–Mills equations with self (or anti-self) dual curvature; in fact, self duality is equivalent to a set of first order differential equations. Atiyah, Drinfeld, Hitchin, Manin [1] demonstrated a construction for any self or anti-self dual finite action connection. An important open question in the classical theory is whether there exist finite action solutions to the second order equations which are not solutions to the first order equations [2]. Insight may be obtained by answering this question in simpler models. In particular, we shall study the O(3) symmetric, SU(2) Yang–Mills equations on $\mathbb{R}^4$ and the Ginzburg–Landau equations [3] with critical coupling constant in two dimensions. The Ginzburg–Landau equations are related to the four dimensional Yang–Mills equations because any O(3) symmetric solution to the SU(2) Yang–Mills equations on the space $\mathbb{R}^2 \times S^2$ with the natural Riemannian metric determines a solution to the Ginzburg–Landau equations and vice versa, c.f. Sect. V.

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