Commun. Math. Phys. 75, 85–101 (1980)

Communications in Mathematical Physics

Positivity and Monotonicity Properties of C_0-Semigroups. II

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Abstract. If $S_t = \exp\{-tH\}$, $T_t = \exp\{-tK\}$, are self-adjoint positivity preserving semigroups on a Hilbert space $\mathscr{H} = L^2(X; d\mu)$ we write

$$T_t \succ 0 \qquad (*)$$

if T_t is positivity improving and

$$S_t \succ T_t \qquad (**)$$

if the difference $S_t - T_t$ is positivity improving. We derive a variety of characterizations of $(*)$ and $(**)$. In particular $(*)$ is valid for all $t > 0$ if, and only if, $T_t \cup L^\infty(X; d\mu)$ is irreducible for some $t > 0$. Similarly if the semigroups are ordered the strict order $(**)$ is valid if, and only if, $\{S_t - T_t\} \cup L^\infty(X; d\mu)$ is irreducible for some $t > 0$. These criteria are used to prove that if $(*)$ is valid for all $t > 0$ then

$$e^{-tf(K)} \succ 0, \quad t > 0,$$

and if $(**)$ is valid for all $t > 0$ then

$$e^{-tf(H)} \succ e^{-tf(K)}, \quad t > 0$$

for each non-constant f in the class characterized in the preceding paper.

We discuss the decomposition of positivity preserving semigroups in terms of positivity improving semigroups on subspaces. Various applications to monotonicity properties of Green's functions are given.

Introduction

A bounded operator A on the Hilbert space $\mathscr{H} = L^2(X; d\mu)$ is called positivity improving if

$$(\phi, A\psi) > 0$$

* On leave of absence from Yokohama City University

0010-3616/80/0075/0085/$03.40