Commun. Math. Phys. 75, 67–84 (1980)

Communications in Mathematical Physics

Positivity and Monotonicity Properties of C_0-Semigroups. I

Ola Bratteli*, Akitaka Kishimoto**, and Derek W. Robinson

Department of Pure Mathematics, University of New South Wales, Kensington 2033, Sydney, Australia

Abstract. If $\exp\{-tH\}$, $\exp\{-tK\}$, are self-adjoint, positivity preserving, contraction semigroups on a Hilbert space $\mathscr{H}=L^2(X;d\mu)$ we write

$$e^{-tH} \geqslant e^{-tK} \geqslant 0 \qquad (*)$$

whenever $\exp\{-tH\}-\exp\{-tK\}$ is positivity preserving for all $t \geqq 0$ and then we characterize the class of positive functions for which $(*)$ always implies

$$e^{-tf(H)} \geqslant e^{-tf(K)} \geqslant 0 .$$

This class consists of the $f \in C^\infty(0,\infty)$ with

$$(-1)^n f^{(n+1)}(x) \geqq 0 , \quad x \in (0,\infty) , \quad n=0,1,2,\dots .$$

In particular it contains the class of monotone operator functions. Furthermore if $\exp\{-tH\}$ is $L^p(X;d\mu)$ contractive for all $p \in [1,\infty]$ and all $t>0$ (or, equivalently, for $p=\infty$ and $t>0$) then $\exp\{-tf(H)\}$ has the same property. Various applications to monotonicity properties of Green's functions are given.

A bounded operator A on the Hilbert space $\mathscr{H}=L^2(X;d\mu)$ is called positivity preserving if

$$(\phi, A\psi) \geqq 0$$

for all non-negative ϕ, ψ, and if this is the case we write

$$A \geqslant 0 .$$

More generally if $A, B, A-B$, are positivity preserving we write

$$A \geqslant B \geqslant 0 .$$

* On leave of absence from the University of Oslo

** On leave of absence from Yokohama City University

0010-3616/80/0075/0067/$03.60