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# Perturbation Theory of Odd Anharmonic Oscillators

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**Abstract.** We study the perturbation theory for $H=p^2+x^2+\beta x^{2n+1}$, $n=1,2,\ldots$. It is proved that when $\mathrm{Im}\beta \neq 0$, $H$ has discrete spectrum. Any eigenvalue is uniquely determined by the (divergent) Rayleigh-Schrödinger perturbation expansion, and admits an analytic continuation to $\mathrm{Im}\beta=0$ where it can be interpreted as a resonance of the problem.

## I. Introduction

It is our purpose in the present paper to study the perturbation theory of the quantum mechanical Hamiltonian $p^2+x^2+\beta x^3$ or, more generally, of any odd anharmonic oscillator $p^2+x^2+\beta x^{2n+1}$, $n=1,2,\ldots$. Unlike the even anharmonic oscillators, extensively studied in recent years (see e.g. Reed and Simon [16] for a review), to our knowledge the above problems did not receive so far any rigorous treatment, in spite of the fact that they are quoted in many textbooks among the simplest examples of bound state perturbation theory (see e.g. Davydov [5]). This is of course due to the well known fact that the Schrödinger operators corresponding to the above Hamiltonians admit infinitely many self-adjoint extensions when defined on $D(p^2)\cap D(x^{2n+1})$. As is known, this leads to the non-uniqueness of the quantum dynamics (for a discussion on this point, and on the connection with the behaviour of the corresponding classical motion, see e.g. Reed and Simon [15]). Although any self-adjoint extension has discrete spectrum, we shall see that the Rayleigh-Schrödinger perturbation theory near any single eigenvalue of $p^2+x^2$ exists (in the sense that the expansion is finite order by order, although divergent) but is related to the self-adjoint extensions only through a spectral concentration phenomenon.

The present situation is thus closely analogous to the Hydrogen Stark effect (with the additional complication of the non-uniqueness of the self-adjoint extensions) where the difficulty has been overcome by showing the existence of

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