Abelian Faces of State Spaces of C^*-Algebras

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Abstract. Let F be a closed face of the weak* compact convex state space of a unital C^*-algebra A. The class of F-abelian states, introduced earlier by the author, is studied further. It is shown (without any restriction on A or F) that F is a Choquet simplex if and only if every state in F is F-abelian, and that it is sufficient for this that every pure state in F is F-abelian. As a corollary, it is deduced that an arbitrary C^*-dynamical system (A, G, α) is G-abelian if and only if every ergodic state is weakly clustering. Nevertheless the set of all F-abelian (or even G-abelian) states is not necessarily weak* compact.

1. Introduction

In the algebraic model of quantum statistical mechanics, decompositions of the invariant states of a C^*-dynamical system (A, G, α) into ergodic states have become important [5]. Particular interest has centered on the question of whether the weak* compact convex set $S_G(A)$ of invariant states forms a Choquet simplex. Lanford and Ruelle [11] showed that this is the case if every invariant state ϕ is G-abelian in the sense that the restriction of $\pi_\phi(A)''$ to the subspace $\mathcal{H}_\phi^G$ of $u_\phi(G)$-invariant vectors in $\mathcal{H}_\phi$ is an abelian von Neumann algebra [where $(\mathcal{H}_\phi, \pi_\phi, u_\phi)$ is the covariant representation of (A, G, α) associated with ϕ]. (This fact was already implicit in [10].) The converse of this result was subsequently obtained by Dang-Ngoc and Ledrappier [7]. Meanwhile it had also been established that for an ergodic state ϕ, G-abelianness is equivalent to the "weak cluster property", namely

$$\inf\{|\phi(a'b) - \phi(a)\phi(b)|\} = 0$$

for all a and b in A, where the infimum is taken over all a' in the convex hull of the G-orbit of a. This raised the question whether every invariant state is G-abelian if every ergodic state is weakly clustering. Dang-Ngoc [6] used direct integral theory to establish this when A is separable.

Recently the present author [4], interested in the class $S_0(A, \alpha)$ of ground states associated with a (strongly continuous) one-parameter C^*-dynamical system