Commun. Math. Phys. 74, 141–188 (1980)

Communications in Mathematical Physics

Nonlinear Wave Equations and Constrained Harmonic Motion

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1. Introduction

Moser and Trubowitz [7] showed that the study of the Korteweg-de Vries equation is simply the study of constrained harmonic motion. Here we show the same is true for the nonlinear Schrödinger, sine-Gordon and Toda lattice equations. Briefly, we have found a change of variables under which these integrable wave equations become a system of free oscillators constrained to an intersection of quadrics in phase space. This is in analogy with the linear case in which the study of wave equations with constant coefficients is reduced via the Fourier transform to the study of harmonic oscillators with linear constraints. One wave equation differs from another only in the nature of the constraint and in each case the constrained system is itself integrable.

In another paper we will use the constrained particle systems to analyze the global phase space geometry of the nonlinear Schrödinger and Sine-Gordon equations. We have no doubt that our technique can be directly applied to the continuous Heisenberg spin chain, the generalized Sine-Gordon equation, the classical Thirring model and any other nonlinear wave equation associated with a second order linear problem.

2. Hamiltonian Mechanics with Constraints[1]

Let $\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$ be the standard symplectic form an R^{2n} and $\{F, G\}$ the corresponding Poisson bracket between smooth functions F and G. If $H \in C^\infty(R^{2n})$ we denote by V_H the corresponding Hamiltonian vector field for which $dH(\cdot) = \omega(V_H, \cdot)$. As usual $S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and $\omega(u, v) = (u, Sv)$.

* Research supported by National Science Foundation Grant No. NSF-MCS-76-07039

** Research supported by National Science Foundation Grant No. NSF-MCS-76-18222

1 For a related discussion of mechanics with constraints see [2]

0010-3616/80/0074/0141/$09.60