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Group-Theoretical Interpretation of the Korteweg-de Vries Type Equations

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Abstract. The Korteweg-de Vries equation is studied within the group-theoretical framework. Analogous equations are obtained for which the many-dimensional Schrödinger equation (with nonlocal potential) plays the same role as the one-dimensional Schrödinger equation does in the theory of the Korteweg-de Vries equation.

1. Introduction

Let $\mathscr{G}$ be an arbitrary Lie algebra, e_i be a basis in $\mathscr{G}$, C_{ij}^k be the corresponding structure constants, $\tilde{\mathscr{G}}$ be the space of linear functionals on $\mathscr{G}$. Denote as $\tilde{e}^i$ the basis in $\tilde{\mathscr{G}}$ which is dual to e_i and as x_i the coordinates in $\tilde{\mathscr{G}}$ with respect to the basis $\tilde{e}^i$.

In the space $\mathscr{F}$ of infinitely-differentiable functions on $\tilde{\mathscr{G}}$ consider the operation

$$[f,g]_{\text{P.B.}} \equiv \{f,g\} = \sum C_{jk}^i x_i \partial^j f \cdot \partial^k g, \qquad \partial^j = \partial/\partial x_j. \tag{1}$$

It was shown in [1] (see also [2–4]) that the operation (1) turns the space of the infinitely-differentiable functions on $\tilde{\mathscr{G}}$ into the Lie algebra. It is natural to call it the Poisson bracket algebra associated with the Lie algebra $\mathscr{G}$.

Let $x = \sum x_i \tilde{e}^i \in \tilde{\mathscr{G}}$, $y = \sum y^i e_i \in \mathscr{G}$,

$$\langle x, y \rangle = \sum x_i y^i. \tag{2}$$

To each function $f(x) \in \mathscr{F}$ and to each $x \in \tilde{\mathscr{G}}$ the element $\nabla f(x) \in \mathscr{G}$ is put into correspondence according to the relation

$$\left.\frac{d}{dt} f(x+ty)\right|_{t=0} = \langle y, \nabla f \rangle = \sum y_i \partial^i f, \qquad y \in \tilde{\mathscr{G}}. \tag{3}$$

With the use of the mapping ∇ the Poisson bracket (1) may be rewritten in the coordinate-independent form

$$\{f(x), g(x)\} = \langle x, [\nabla f, \nabla g] \rangle, \tag{4}$$

where $[\nabla f(x), \nabla g(x)]$ stands for the commutator in $\mathscr{G}$.

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