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The Boltzmann Equation with a Soft Potential

II. Nonlinear, Spatially-Periodic

Russel E. Caflisch*

Department of Mathematics, Stanford University, Stanford, CA 94305, USA

Abstract. The results of Part I are extended to include linear spatially periodic problems–solutions of the initial value are shown to exist and decay like $e^{-\lambda t^{\beta}}$. Then the full non-linear Boltzmann equation with a soft potential is solved for initial data close to equilibrium. The non-linearity is treated as a perturbation of the linear problem, and the equation is solved by iteration.

1. Introduction

The linear Boltzmann equation with a soft intermolecular potential was solved globally in time in Part I [1], if the initial density is a spatially homogeneous perturbation of a global Maxwellian. Moreover it was proven that this perturbation decays in $\mathscr{L}^2$ or sup norm like $e^{-\lambda t^{\beta}}$, with $\lambda>0$, $1>\beta>0$, if it is initially bounded by a Maxwellian. We will refer to formulas or results from Part I by preceeding their numbers with an "I" as in (I 1.7).

In this paper we find the same result even if the initial perturbation is *spatially dependent* in the cube with periodic boundary conditions. In addition we can solve the spatially periodic *nonlinear* problem globally in time if the initial perturbation is small enough, and we find that the solution decays to the Maxwellian equilibrium.

The linear, spatially-dependent Boltzmann equation is

$$\frac{\partial}{\partial t} f + \boldsymbol{\xi} \cdot \frac{\partial}{\partial \mathbf{x}} f + Lf = 0, \tag{1.1}$$

$$f(t=0) = f_0 \in \mathscr{N}, \tag{1.2}$$

where f_0 and $f = f(t, \mathbf{x}, \boldsymbol{\xi})$ are periodic in $\mathbf{x} \in T^3 = [0, 2\pi]^3$, $t \geqq 0$, $\boldsymbol{\xi} \in \mathbb{R}^3$, and $\mathscr{N} = \left\{ g(\mathbf{x}, \boldsymbol{\xi}) : \int_{T^3} \int_{\mathbb{R}^3} \psi(\boldsymbol{\xi}) g(\mathbf{x}, \boldsymbol{\xi})\, d\boldsymbol{\xi}\, d\mathbf{x} = 0 \text{ for } \psi(\boldsymbol{\xi}) = 1, \xi_i, \text{ or } \xi^2 \right\}$. The requirement that $f_0 \in \mathscr{N}$ just means that we have chosen the right Maxwellian equilibrium to

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