Commun. Math. Phys. 72, 103–118 (1980)

Communications in
**Mathematical
Physics**

Generalized Spin Structures on Four Dimensional Space-Times

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Abstract. We discuss various methods for investigating the existence and uniqueness of generalized spin structures. We show that on a four dimensional manifold whole families may be constructed using any internal symmetry group of the form $G/\mathbb{Z}_2$, where G is a simply connected Lie group.

1. Introduction

Much of the current work in quantum gravity is concerned with the construction of classical and quantized fields propagating on a fixed background space-time M. This subject is of interest in its own right and also occurs in the evaluation of the one loop contribution to a functional integral approach to quantum gravity proper. An especially intriguing aspect is the rôle played by the global topology of the space-time.

In the present paper we will concentrate on aspects of space-time topology that are reflected in spinor field theory. There will in general be a number of inequivalent spinor structures [13] and these are classified by elements of the cohomology group $H^1(M;\mathbb{Z}_2)$. We have previously discussed a number of features of this phenomenon relevant to quantum field theory [12], [2]. However the very existence of spinors is determined by global topological properties of M; specifically a necessary and sufficient condition is the vanishing of the second Stiefel Whitney class of the tangent bundle [13]. The problem of handling spacetimes which do not satisfy this restriction has been approached in various ways. We wish to focus on the ideas of Whiston [18], Hawking et al. [8], Back et al. [4], and Forger et al. [7], and discuss and extend their techniques. Hawking and Pope sought to replace the Spin(4) group covering SO(4) with $\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\mathbb{Z}_2} U(1)$ whilst Back et al. employed $\mathrm{Spin}(4) \times_{\mathbb{Z}_2} G$ where G was basically SU(2). The $\mathrm{Spin}^c(4)$ method is limited to a special class of space-times whereas the construction of Back et al, although free of this defect, represents only one of a large class of $\mathrm{Spin}(4) \times_{\mathbb{Z}_2} G$ covering techniques. We will develop some of these in Sec. 4 and to motivate and clarify the method employed we will first discuss the existing work in Sect. 2 and 3 using the appropriate mathematical language.