Commun. Math. Phys. 72, 49–53 (1980)

Communications in Mathematical Physics

On a Normal Form of Symmetric Maps of [0, 1]

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Abstract. A class of continuous symmetric mappings of [0, 1] into itself is considered leaving invariant a measure absolutely continuous with respect to the Lebesgue measure.

We consider a continuous map f of the closed unit interval onto itself and try to put it into the normal form $N=\varphi^{-1}\circ f\circ\varphi$,

$$N(x)=\begin{cases}2x & \text{if} \quad 0\leqq x\leqq \frac{1}{2}\\ 2(1-x) & \text{if} \quad \frac{1}{2}\leqq x\leqq 1\end{cases} \tag{1}$$

by means of a homeomorphism φ of $[0,1]$. The statement is as follows:

Theorem. *Let $f:[0,1]\to[0,1]$ be continuous and satisfying*

$$f(0)=0,\, f(\tfrac{1}{2})=1\,, \tag{2}$$

$$f(x)=f(1-x), \quad 0\leqq x\leqq 1\,. \tag{3}$$

Assume that

$$1<c\leqq\frac{f(x)-f(y)}{x-y}, \quad 0\leqq y<x\leqq\tfrac{1}{2} \tag{4}$$

for a real constant c. Then there is a strictly increasing continuous map φ of $[0,1]$ onto itself such that

$$\varphi(Nx)=f(\varphi(x)), \quad 0\leqq x\leqq 1 \tag{5}$$

with N as defined by (1). *Moreover, if $c>2^\sigma$ for some σ with $0<\sigma<1$ then φ is Hölder-continuous with exponent σ.*

Remark 1. Observe the condition (4) is essentially a smallness condition on the Lipschitz distance between the functions $x\to f(x)$ and $x\to 2x$, $0\leqq x\leqq\frac{1}{2}$.

0010-3616/80/0072/0049/$01.00