Commun. Math. Phys. 71, 299–309 (1980)

Communications in
Mathematical
Physics

Analyticity of Solutions of the $O(N)$ Nonlinear σ-Model

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Abstract. We consider continuous weak solutions of the Euler-Lagrange equations associated with the Euclidean d-dimensional $O(N)$ nonlinear σ-model. We show for arbitrary N and arbitrary d that such solutions with locally square integrable gradient are real analytic.

1. Introduction

We consider solutions of the d-dimensional ($d \geqq 2$) Euclidean $O(N)$ non-linear σ-model, i.e. stationary points of the Lagrangian

$$L(n) = \sum_{\alpha=1}^{d} \sum_{l=1}^{N} (\partial_\alpha n_l)^2 \equiv (\nabla n)^2 \tag{1.1}$$

where $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$, and $n \in \mathbb{R}^N$ satisfies the constraint

$$n^2 \equiv |n|^2 \equiv (n, n) := \sum_{l=1}^{N} n_l^2 = 1 \,. \tag{1.2}$$

Stationary points n of L such that $L(n)$ is locally L^1 are (weak) solutions of the Euler-Lagrange equations associated with (1.1)

$$\Delta n_l + L(n) n_l = 0 \qquad l = 1 \dots N \,. \tag{1.3}$$

(A detailed proof of this fact along the lines of the usual variational argument has been given in [1] where also the class of variations was specified.)

Since the left hand side of (1.3) is an elliptic operator, one may expect weak solutions to show some regularity, i.e. to be C^k (k times continuously differentiable) for some k. There is an extensive literature on elliptic regularity, and we quote only some results relevant for (1.3):

In 1929, Lewy [2] gave a lucid proof of Bernstein's theorem that in two dimensions ($d = 2$), every C^3-solution of a nonlinear elliptic equation with analytic

0010-3616/80/0071/0299/$02.20