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The Classical Limit of Quantum Partition Functions

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Abstract. We extend Lieb's limit theorem [which asserts that SO(3) quantum spins approach S^2 classical spins as $L\to\infty$] to general compact Lie groups. We also discuss the classical limit for various continuum systems. To control the compact group case, we discuss coherent states built up from a maximal weight vector in an irreducible representation and we prove that every bounded operator is an integral of projections onto coherent vectors (i.e. every operator has "diagonal form").

1. Introduction

This paper is motivated in the first place by a beautiful paper of Lieb [23] who considers the following situation. Let Λ be a finite set and let $H(\boldsymbol{S}_\alpha)$ be a function of $R^{3|\Lambda|}$ variables $\{S_{\alpha,i}\}$, $\alpha\in\Lambda$, $i=1, 2, 3$ which is multiaffine, i.e. a sum of monomials which are of degree zero or one in the variables at each site. Define

$$Z_{c\ell}(\gamma)=\int\prod_{\alpha\in\Lambda}[d\Omega(S_\alpha)/4\pi]\exp(-H(\gamma S_\alpha)) \tag{1.1}$$

where $d\Omega$ is the usual (unnormalized) measure on the unit sphere, S^2, in R^3. For each $\ell=1/2, 1, 3/2, \ldots$, let

$$Z_Q^\ell(\gamma)=(2\ell+1)^{-|\Lambda|}\operatorname{Tr}(\exp[-H(\gamma L_\alpha/\ell)]) \tag{1.2}$$

where $\{L_\alpha\}$ is a family of independent spin ℓ quantum spins, i.e. L_α acts on $(C^{2\ell+1})^{|\Lambda|}$ thought of as a tensor product with $L_\alpha=1\otimes\ldots\otimes\underset{\sim}{L}\otimes\ldots\otimes 1$ (not 1 only in the αth factor) and $\underset{\sim}{L}$ the usual vector of angular momentum ℓ. Then Lieb [23] proves:

$$Z_{c\ell}(\gamma)\leqq Z_Q^\ell(\gamma)\leqq Z_{c\ell}(\gamma+\ell^{-1}\gamma). \tag{1.3}$$

This demonstrates convergence of Z_Q to $Z_{c\ell}$ as $\ell\to\infty$ in a sufficiently strong way that one can interchange the $\ell\to\infty$ and the $|\Lambda|\to\infty$ limit in the free energy per unit volume.

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