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# Semiclassical Quantum Mechanics

## I. The $\hbar \to 0$ Limit for Coherent States

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**Abstract**. We consider the $\hbar \to 0$ limit of the quantum dynamics generated by the Hamiltonian $H(\hbar) = -(\hbar^2/2m)\Delta + V$. We prove that the evolution of certain Gaussian states is determined asymptotically as $\hbar \to 0$ by classical mechanics. For suitable potentials $V$ in $n \geqq 3$ dimensions, our estimates are uniform in time and our results hold for scattering theory.

## 1. Introduction and Results

The purpose of this paper is to give simple proofs of two theorems concerning the classical limit of quantum mechanics. Our first theorem deals with finite times. The second deals with infinite times in the framework of scattering theory.

For the potentials which we consider, the quantum Hamiltonian $H(\hbar) = -(\hbar^2/2m)\Delta + V(x) = H_0(\hbar) + V$ on $L^2(\mathbb{R}^n)$ is essentially self-adjoint on the $C^\infty(\mathbb{R}^n)$ functions of compact support. The corresponding classical Hamiltonian is $H(\xi, x) = \xi^2/2m + V(x)$ on the phase space $\mathbb{R}^{2n}$.

Under the Hamiltonian $H(\hbar)$ we study the evolution of the Gaussian wave functions $\psi_\alpha(A, B, \hbar, a, \eta, x)$ which are defined below. The state $\psi_\alpha(A, B, \hbar, a, \eta, x)$ is concentrated near the position $a$ and near the momentum $\eta$. Heuristically, its position width and momentum width are given by the matrices $\hbar^\alpha (AA^*)^{1/2}$ and $(\hbar^{1-\alpha}/2)(BB^*)^{1/2}$, respectively. Our motivation for considering the states $\psi_\alpha(A, B, \hbar, a, \eta, x)$ comes from explicit calculations for harmonic oscillators. The precise definition is the following:

*Definition.* Let $A$ and $B$ be complex $n \times n$ matrices with the following properties:

$A$ and $B$ are invertible; (1.1)

$BA^{-1}$ is symmetric ((real symmetric) + $i$ (real symmetric)); (1.2)

$\mathrm{Re}\, BA^{-1} = \frac{1}{2}[(BA^{-1}) + (BA^{-1})^*]$ is strictly positive definite; (1.3)

$(\mathrm{Re}\, BA^{-1})^{-1} = AA^*$. (1.4)

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