Commun. Math. Phys. 71, 65–76 (1980)

Communications in
Mathematical
Physics

Differentiability Properties of the Pressure in Lattice Systems

H. A. M. Daniëls and A. C. D. van Enter

Institute for Theoretical Physics, University of Groningen,
P.O. Box 800, Groningen, Netherlands

Abstract. In two recent papers Ruelle gave a heuristic theory of phase transitions, using some techniques introduced by Israel. He proves a version of the Gibbs phase rule, assuming a differentiability condition for the pressure. Ruelle already pointed out that his condition cannot always hold. In this paper we prove that the interaction spaces which he considers are in general too large for his condition to hold. We also show that the version of the Gibbs phase rule which is a consequence of this condition does not hold in general. Moreover we give some constraints on the analyticity properties of the pressure.

1. Introduction

In two recent papers Ruelle [1, 2] proposed a heuristic theory of phase transitions. He shows that every interaction which admits n phases lies in a manifold of codimension $n-1$ of interactions which admit n or more phases, if the pressure is differentiable in a certain sense.

In this paper we will study his differentiability condition. We will firstly show that on the usual space of interactions of which the pressure is defined, the condition never holds. In the second part of our paper we consider a smaller space of interactions where it is possible to discriminate between differentiability (and also analyticity) properties at low and high temperatures. We will prove that in a more phase region the pressure is not Fréchet differentiable and therefore not analytic in the space of pair interactions (Theorem 1) (for 1-dimensional systems this result was proven by Ruelle [2]). Moreover we show that, in general, spaces of the type considered by Ruelle are too large to obtain manifolds of more phases and that the version of the Gibbs phase rule as proposed in [2] cannot be true (Theorem 2).

We follow Israel considering a quantum lattice (the same results hold for classical lattices).

I. We consider a lattice $\mathbb{Z}^\nu$. At each point $x \in \mathbb{Z}^\nu$, there is defined an identical m-dimensional Hilbert space $\mathfrak{H}_x$. For each finite subset X of $\mathbb{Z}^\nu$ the Hilbert space $\mathfrak{H}_X = \bigotimes_{x \in X} \mathfrak{H}_x$ is defined.

0010-3616/80/0071/0065/$02.40