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On the Euclidean Approach to Quantum Field Theory in Curved Spacetime*

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Abstract. Quantum field theory in curved spacetime is examined from the Euclidean approach, where one seeks to define the theory for metrics of positive (rather than Lorentzian) signature. Methods of functional analysis are used to give a proof of the heat kernel expansion for the Laplacian, which extends the well known result for compact manifolds to all non-compact manifolds for which the Laplacian and its powers are essentially self-adjoint on the initial domain of smooth functions of compact support. Using this result, precise prescriptions of the zeta-function, dimensional, and point-splitting type are given for renormalizing the action of a Klein-Gordon scalar field. These prescriptions are shown to be equivalent up to local curvature terms. It is also shown that for static spacetimes, the Euclidean prescription for defining the Feynman propagator agrees with the definition of Feyman propagator obtained by working directly on the spacetime.

I. Introduction

In quantum field theory, it is common to derive in a formal way expressions for quantities of physical interest. Taken literally, these expressions frequently are either meaningless or infinite. Thus, a major task in quantum field theory is to give a well defined, finite meaning to these formal expressions.

The theory of free quantum fields in curved spacetime is sufficiently simple that much of it – for example, the derivation of the S-matrix – can be mathematically formulated in a perfectly well defined manner (see e.g. [1]). However, even in this simple case there are two major problems that confront one: (1) the definition of "in" and "out" particle states (or, equivalently, the definition of the Feynman propagator) in general circumstances – in particular, for spacetimes with initial and/or final singularities; (2) the definition of finite, mathematically well defined expressions for physical quantities whose formal expressions are nonlinear in the

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